Section 4.5
A second example

Math F251X: Calculus 1

Example: $j(x) = \dfrac{(1+x)^2}{1+x^2}$

① <u>Domain</u>: Observe $1+x^2 > 0$ for all x

Domain $= \mathbb{R}$

② <u>Intercepts</u>

y-intercept: $j(0) = \dfrac{(1+0)^2}{1+0^2} = 1$

x-intercept: solve $j(x) = 0 \Rightarrow \dfrac{(1+x)^2}{1+x^2} = 0$

$$\Rightarrow (1+x)^2 = 0$$

$$\Rightarrow x = -1$$

So our curve goes through $(0,1)$ and $(-1,0)$.

$$j(x) = \frac{(1+x)^2}{1+x^2}$$

Asymptotes

- No vertical asymptotes
- Horizontal asymptotes:

$$\lim_{x\to\infty} j(x) = \lim_{x\to\infty} \frac{1+2x+x^2}{1+x^2}$$

$$= \lim_{x\to\infty} \frac{1/x^2 + 2/x + 1}{1/x^2 + 1}$$

$$= 1$$

$$\lim_{x\to-\infty} j(x) = \lim_{x\to\infty} \frac{1+2(-x)+(-x)^2}{1+(-x)^2}$$

$$= \lim_{x\to\infty} \frac{1-2x+x^2}{1+x^2}$$

$$= \lim_{x\to\infty} \frac{1/x^2 - 2/x + 1}{1/x^2 + 1}$$

$$= 1$$

So $y=1$ is a horizontal asymptote in both directions!

Increasing / Decreasing / Max / Min

$$j(x) = \frac{(1+x)^2}{1+x^2} \Rightarrow j'(x) = \frac{(1+x^2)(2(1+x)) - (1+x)^2(2x)}{(1+x^2)^2}$$

$$= \frac{2(1+x)((1+x^2) - (1+x)x)}{(1+x^2)^2} = \frac{2(1+x)(1+x^2-x-x^2)}{(1+x^2)^2} = \frac{2(1+x)(1-x)}{\boxed{(1+x^2)^2}}$$

always positive!

$$j'(x) = 0 \Rightarrow 2(1+x)(1-x) = 0 \Rightarrow x = 1 \text{ or } x = -1$$

$j'(x)$ is never undefined!

x		-1		1	
test	-2		0		2
Sign f'	−	0	+	0	−
f					

$$f'(-2) = \frac{2(1-2)(1+2)}{+} = \frac{2(-)(+)}{+} = -$$

$$f'(0) = \frac{2(1+0)(1-0)}{+} = \frac{2(+)(+)}{+} = +$$

$$f'(2) = \frac{2(1+2)(1-2)}{+} = \frac{2(+)(-)}{+} = -$$

Concavity and inflection points

$$j(x) = \frac{(1+x)^2}{1+x^2}$$

$$j'(x) = \frac{2(1+x)(1-x)}{(1+x^2)^2}$$

$$j'(x) = \frac{2(1-x^2)}{(1+x^2)^2}$$

$$\Rightarrow j''(x) = \frac{(1+x^2)^2(2)(-2x) - 2(1-x^2)(2)(1+x^2)(2x)}{(1+x^2)^4}$$

$$= \frac{-2x(1+x^2)\big((1+x^2)(2) + (1-x^2)(2)(2)\big)}{(1+x^2)^4} = \frac{-2x\cancel{(1+x^2)}(2+2x^2+4-4x^2)}{(1+x^2)^{\cancel{4}\,3}}$$

$$= \frac{-2x(6-2x^2)}{(1+x^2)^3} = \frac{-4x(3-x^2)}{(1+x^2)^3}$$

$$j''(x) = 0 \Rightarrow -4x(3-x^2) = 0 \Rightarrow \boxed{x=0} \text{ or } x^2 = 3 \Rightarrow \boxed{x=\sqrt{3}} \text{ or } \boxed{x=-\sqrt{3}}$$

$j''(x)$ is never undefined!

Recall $1 < 3 < 4 \Rightarrow 1 < \sqrt{3} < 2$

$$j''(-2) = \frac{-4(-2)(3-4)}{+} = \frac{(-)(-)(-)}{+} \qquad j(1) = \frac{(-)(+)+}{+}$$

$$j''(-1) = \frac{-4(-1)(3-1)}{+} = \frac{(-)(-)(+)}{+} \qquad j(2) = \frac{-(+)(-)}{+}$$

x		$-\sqrt{3}$		0		$\sqrt{3}$	
test	-2		-1		1		2
j''	$-$	0	$+$	0	$-$	0	$+$
j	$\cap$		$\cup$		$\cap$		$\cup$

$$j(x) = \frac{(1+x)^2}{1+x^2}$$

domain = $\mathbb{R}$

HA at $y = +1$

x		-1		1	
j'	↘	MIN ✓	↗	MAX ✓	↘

x		$-\sqrt{3}$		0		$\sqrt{3}$	
j''	$\cap$ ✓	INF	$\cup$ ✓	INF	$\cap$ ✓	INF ✓	$\cup$